Exact renormalization group approach to a nonlinear diffusion equation

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The exact renormalization group is applied to a nonlinear diffusion equation with a discontinuous diffusion coefficient. The generating functional of the solution for the initial-value problem of nonlinear diffusion equations is first introduced, and next a regularization scheme is presented. It is shown that the renormalization of an action functional in the generating functional leads to an anomalous diffusion exponent in full order of the perturbation series with respect to a nonlinearity.

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I. INTRODUCTION

The renormalization group (RG) is a powerful tool to reveal universal behavior of various systems, including quantum field theories and statistical mechanics [1]. Its basic idea lies in the coarse-graining of short-distance degrees of freedom, which causes redefinition of parameters governing the long-distance physics of the systems under investigation. In spite of its conceptual simplicity, there exist in the RG method many techniques for solution, and numerous applications have been made to equilibrium or near-equilibrium systems [2,3]. In particular, exact RG (ERG) (also called nonperturbative RG or functional RG) techniques and some approximations based on them [1,4–7] have been attracting much renewed interest in revealing nonperturbative phenomena in field theories [8], statistical mechanics [9], and condensed matter physics [10–12].

On the other hand, Goldenfeld *et al.* [13,14] have extended the RG method to systems far from equilibrium. They have demonstrated that there exists a deep relationship between the RG and the intermediate asymptotics method [15] in the study of the nonlinear partial-differential equations for nonequilibrium systems. Their idea has attracted much interest, and the RG approach to nonlinear differential equations has been developed [16].

In this paper, we apply the ERG method to a nonlinear diffusion equation called the Barenblatt equation [15]. This equation has a discontinuous diffusion coefficient; this discontinuity makes perturbative expansion more complicated if one proceeds to higher order computations. We show in this paper that the Polchinski equation, a version of the ERG equation, is very efficient even for such a nonlinear diffusion equation. It turns out that we can indeed solve the equation *for all orders* in the perturbation series. The solution leads us to the full anomalous diffusion exponent.

The outline of this paper is as follows. In Sec. II, we first introduce the Barenblatt equation, and next define the generating functional of the solution of this equation. We also introduce a regularization scheme convenient for the application of the ERG method. In Sec. III, we derive the Polchinski equation for the Barenblatt equation and solve it by assuming a particular form of the solution. This solution leads us to the anomalous diffusion exponent which we compute in Sec. IV. We summarize our results in Sec. V.

II. BARENBLATT EQUATION

In this section, we introduce the generating functional of the solution for the Barenblatt equation and a regularization scheme to render the solution finite. These play an important role in the application of ERG techniques to the present system.

Let us start with the following nonlinear diffusion equation called the Barenblatt equation:

$$\partial_t u(x,t) - D(u)\partial_x^2 u(x,t) = 0, \tag{1}$$

with initial condition

$$u(x,t=0) = q\delta(x), \tag{2}$$

where

$$D(u) \equiv \kappa[1 + g\theta(-\alpha\partial_t u)] \tag{3}$$

denotes a nonlinear diffusion coefficient with α being a positive constant which makes $\alpha\partial_t u$ dimensionless (below, we set $\alpha=1$, for simplicity). Here, $\theta(x)=0$ (1) for $x<0$ ($x>0$) stands for the step function. The dimensionless constant g controls the nonlinearity of the diffusion coefficient. The Barenblatt equation describes the filtration of a compressible fluid through a compressible porous medium which can be irreversibly deformed. Goldenfeld *et al.* [13] have obtained asymptotic behavior of the solution by solving this equation via an iteration scheme corresponding to a perturbative expansion with respect to g. This perturbation gives rise to divergences: Their basic idea is introducing a renormalization scheme which renders the solution finite and deriving an anomalous diffusion exponent as an anomalous dimension in the RG language. Although they have successfully obtained the leading correction of the diffusion exponent, their method seems difficult to extend to higher order due to the discontinuous step-function nonlinearity. In the following sections, we will introduce the generating functional of the solution, and present a regularization scheme for the initial-value problems of nonlinear diffusion equations.

A. Generating functional of the solution

First of all, we introduce the generating functional of the solution for the Barenblatt equation. To this end, notice [3,17] that the solution of Eq. (1) can be written as

$$u(x,t)=\int \mathcal{D}\phi\, \phi(x,t)\left(\prod_{t,x}\delta(\partial_t\phi - D(\phi)\partial_x^2\phi)\right)\times\prod_x \delta(\phi(x,0)-u(x,0)). \quad (4)$$

This expression can be rewritten as a functional integral if the derivative with respect to t is interpreted as a forward difference operator; namely, using the Fourier transformation for the δ function, we reach

$$u(x,t)=\langle\phi(x,t)\rangle=\frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\, \mathcal{D}\tilde{\phi}\, \phi(x,t)e^{iS}, \quad (5)$$

with S being an action functional

$$S=\int_0^\infty dt\int_{-\infty}^{\infty} dx[\tilde{\phi}\Delta^{-1}\phi - g\tilde{\phi}\theta(-\partial_t\phi)\kappa\partial_x^2\phi - \tilde{\phi}J], \quad (6)$$

where

$$\Delta(x,t)\equiv e^{-x^2/(4\kappa t)}/\sqrt{4\pi\kappa t} \quad (7)$$

denotes the diffusion propagator and the generating functional $\mathcal{Z}$ is defined by $\mathcal{Z}\equiv\int\mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\, e^{iS}$, as usual. The field J in the last term is defined by

$$J(x,t)\equiv u(x,0)\delta(t) \quad (8)$$

which controls the initial value of u. In what follows, we examine the case with an initial condition (2), but it should be stressed that the generic initial-value problem can be treated similarly.

B. Regularization scheme

As discussed by Goldenfeld *et al.* [13], the perturbative calculation diverges with the initial condition (2). To regularize the solution, they have introduced an initial distribution with a finite width such as $u(x,t=0)=e^{-x^2/(2l^2)}/\sqrt{2\pi l^2}$. We instead introduce the short-time cutoff ε for the propagator to formulate the ERG for the present nonlinear diffusion equation. To be specific, we define a modified propagator as

$$\Delta_\varepsilon(x,t)=\theta(t-\varepsilon)\Delta(x,t). \quad (9)$$

One can easily check that this propagator indeed gives a finite solution in the perturbation theory, applying it to the calculations by Goldenfeld *et al.* [13]. This regularization scheme can be used not only in the Barenblatt equation but also in generic diffusion problems.

III. ERG EQUATION

Having defined the generating functional and the modified propagator, we next derive an ERG equation for the action functional (6).

A. Derivation of the Polchinski equation

Using the propagator (9) with a cutoff ε_0 and introducing a source term, we start with the generating functional

$$\mathcal{Z}[\tilde{J},J]=\int \mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\, e^{i\tilde{\phi}\cdot\Delta_{\varepsilon_0}^{-1}\cdot\phi - iS_{\varepsilon_0}[\tilde{\phi},\phi]-i\tilde{J}\cdot\phi - i\tilde{\phi}\cdot J}, \quad (10)$$

where the bare action of the nonlinear term is

$$S_{\varepsilon_0}[\tilde{\phi},\phi]=g\tilde{\phi}\cdot\theta(-\partial_t\phi)\kappa\partial_x^2\phi. \quad (11)$$

At the end of the calculations, we must set $J(x,t)=q\delta(x)\delta(t)$ to obtain the solution for the initial-value problem of the present equation. Here, the symbol $a\cdot b$ implies

$$a\cdot b=\int_0^\infty dt\int_{-\infty}^{\infty} dx\, a(x,t)b(x,t). \quad (12)$$

Next, we introduce a new cutoff ε $(>\varepsilon_0)$ and divide the propagator into two parts $\Delta_{\varepsilon_0}=\Delta_>+\Delta_<$, where

$$\Delta_>=[\theta(t-\varepsilon_0)-\theta(t-\varepsilon)]\Delta,$$

$$\Delta_<=\theta(t-\varepsilon)\Delta. \quad (13)$$

Here, $>$ and $<$ imply the short-time and long-time modes, respectively. Separating also ϕ and $\tilde{\phi}$ into two fields $\phi=\phi_>+\phi_<$ and $\tilde{\phi}=\tilde{\phi}_>+\tilde{\phi}_<$ enables us to rewrite the generating functional as

$$\mathcal{Z}[\tilde{J},J]\propto\int \mathcal{D}\phi_<\mathcal{D}\tilde{\phi}_< e^{i\tilde{\phi}_<\cdot\Delta_<^{-1}\cdot\phi_<}\mathcal{Z}_\varepsilon[\tilde{J},J;\tilde{\phi}_<,\phi_<],$$

$$\mathcal{Z}_\varepsilon[\tilde{J},J;\tilde{\phi}_<,\phi_<]=\int\mathcal{D}\phi_>\mathcal{D}\tilde{\phi}_> e^{i\tilde{\phi}_>\cdot\Delta_>^{-1}\cdot\phi_>}\times e^{-iS_{\varepsilon_0}[\tilde{\phi}_>+\tilde{\phi}_<,\phi_>+\phi_<]-i\tilde{J}\cdot(\phi_>+\phi_<)-i(\tilde{\phi}_>+\tilde{\phi}_<)\cdot J}, \quad (14)$$

up to a proportionality constant. The field $\phi_>$, $\tilde{\phi}_>$ and $\phi_<,\tilde{\phi}_<$ can be identified as fields describing short-time and long-time modes, respectively. Integrating out the short-time fields, we will next derive an effective action describing long-time modes.

Changing the integration variables $\tilde{\phi}_>$ and $\phi_>$ into $\tilde{\phi}_>=\tilde{\phi}-\tilde{\phi}_<$ and $\phi_>=\phi-\phi_<$, and integrating over the fields $\tilde{\phi}$ and ϕ in Eq. (14) yields

$$\mathcal{Z}_\varepsilon[\tilde{J},J;\tilde{\phi}_<,\phi_<]=\int\mathcal{D}\tilde{\phi}\,\mathcal{D}\phi\, e^{i(\tilde{\phi}-\tilde{\phi}_<)\cdot\Delta_>^{-1}\cdot(\phi-\phi_<)-iS_{\varepsilon_0}[\tilde{\phi},\phi]-i\tilde{J}\cdot\phi-i\tilde{\phi}\cdot J}=e^{-i\tilde{J}\cdot\Delta_>\cdot J-i\tilde{\phi}_<\cdot J-i\tilde{J}\cdot\phi_<-iS_\varepsilon[\tilde{J}\cdot\Delta_>+\tilde{\phi}_<,\Delta_>\cdot J+\phi_<]}, \quad (15)$$

where S_ε is defined by

$$e^{-iS_\varepsilon[\tilde{J}\cdot\Delta_> + \tilde{\phi}_<, \Delta_>\cdot J+\phi_<]} \equiv e^{-iS_{\varepsilon_0}[i\delta/\delta J + \tilde{J}\cdot\Delta_> + \tilde{\phi}_<, i\delta/\delta\tilde{J} + \Delta_>\cdot J + \phi_<]}. \tag{16}$$

This equation implies that if we expand the exponential in the right-hand side (RHS) and make all the derivatives $\delta/\delta J$ and $\delta/\delta\tilde{J}$ in S_{ε_0} act on J and $\tilde{J}$ in the right S_{ε_0}, we reach some S_ε as a functional of

$$\tilde{\Phi} = \tilde{J}\cdot\Delta_> + \tilde{\phi}_<,$$

$$\Phi = \Delta_>\cdot J + \phi_<. \tag{17}$$

Instead of carrying out such calculations, however, we can alternatively determine the functional S_ε by noting that $\mathcal{Z}_\varepsilon$ obeys

$$\frac{d\mathcal{Z}_\varepsilon}{d\varepsilon} = i\left(i\frac{\delta}{\delta J} - \tilde{\phi}_<\right)\cdot\frac{d\Delta_>^{-1}}{d\varepsilon}\cdot\left(i\frac{\delta}{\delta\tilde{J}} - \phi_<\right)\mathcal{Z}_\varepsilon, \tag{18}$$

which follows from Eq. (14). Substituting Eq. (15) into Eq. (18), we obtain the following Polchinski RG equation:

$$\frac{\partial S_\varepsilon}{\partial\varepsilon} = \frac{\delta S_\varepsilon}{\delta\Phi}\cdot\frac{d\Delta_>}{d\varepsilon}\cdot\frac{\delta S_\varepsilon}{\delta\tilde{\Phi}} + i\,\mathrm{tr}\frac{d\Delta_>}{d\varepsilon}\cdot\frac{\delta^2 S_\varepsilon}{\delta\tilde{\Phi}\,\delta\Phi}. \tag{19}$$

This equation can be simplified by taking into account the $\tilde{\Phi}$ dependence of the functional S_ε. The bare S_{ε_0} contains only the first order term in $\tilde{\Phi}$, but in the process of the renormalization, Eq. (19) yields the zeroth order term in S_ε. To be concrete, let us denote

$$S_\varepsilon[\tilde{\Phi},\Phi] = \tilde{\Phi}\cdot H_\varepsilon[\Phi] + F[\Phi]. \tag{20}$$

Substituting this into Eq. (19), we find

$$\frac{\partial H_\varepsilon}{\partial\varepsilon} = \frac{\delta H_\varepsilon}{\delta\Phi}\cdot\frac{d\Delta_>}{d\varepsilon}\cdot H_\varepsilon, \tag{21}$$

$$\frac{\partial F_\varepsilon}{\partial\varepsilon} = \frac{\delta F_\varepsilon}{\delta\Phi}\cdot\frac{d\Delta_>}{d\varepsilon}\cdot H_\varepsilon + i\mathrm{tr}\frac{d\Delta_>}{d\varepsilon}\cdot\frac{\delta H_\varepsilon}{\delta\Phi} \tag{22}$$

with the bare functions

$$H_{\varepsilon_0}[\Phi] = g\,\theta(-\partial_t\Phi)\kappa\partial_x^2\Phi,$$

$$F_{\varepsilon_0}[\Phi] = 0. \tag{23}$$

The RG equation for the H_ε term, which determines the solution of the Barenblatt equation, is closed. Furthermore, it has no loop corrections. Nevertheless, the initial-value problems are still nontrivial, since after obtaining H_ε, we must set $J=q\delta(x)\delta(t)$ and determine the ε dependence.

B. Solution of Polchinski equation

Since the functional equation (21) is still too difficult to obtain analytic solutions, we assume the form of H_ε as

$$H_\varepsilon[\Phi](x,t) = \int_0^\infty ds\int_{-\infty}^\infty dy\; V_\varepsilon[\partial_t\Phi(x-y,t-s)]\kappa\partial_y^2\Phi(y,s)$$

$$\equiv V_\varepsilon[\dot{\Phi}]\cdot\kappa\Phi''(x,t) \tag{24}$$

with a certain unknown function V_ε, where we have denoted $\partial_t\Phi=\dot{\Phi}$ and $\partial_x^2\Phi=\Phi''$ for simplicity. This assumption seems natural from the point of view of the derivative expansion, since the bare functional is second order with respect to ∂_x. Substituting this into Eq. (21), we have

$$\partial_\varepsilon V_\varepsilon[\dot{\Phi}] = V_\varepsilon[\dot{\Phi}]\cdot\partial_\varepsilon\kappa\Delta_>''\cdot V_\varepsilon[\dot{\Phi}]. \tag{25}$$

The bare function is given by $V_{\varepsilon_0}[\dot{\Phi}](x-y,t-s) = -g\,\theta(-\dot{\Phi}(x,t))\delta(x-y)\delta(t-s)$. This equation can be solved if V_ε is expanded in power series of g such that

$$V_\varepsilon = \sum_{n=1}^\infty g^n V_\varepsilon^{(n)}. \tag{26}$$

Actually, we find that each term obeys

$$\partial_\varepsilon V_\varepsilon^{(n)}[\dot{\Phi}] = \sum_{n_1+n_2=n} V_\varepsilon^{(n_1)}[\dot{\Phi}]\cdot\partial_\varepsilon\kappa\Delta_>''\cdot V_\varepsilon^{(n_2)}[\dot{\Phi}]. \tag{27}$$

From this equation, it follows that $\partial_\varepsilon V_\varepsilon^{(1)}=0$ and hence, it turns out that $V_\varepsilon^{(1)}$ is not renormalized; that is, $V_\varepsilon^{(1)}[\dot{\Phi}] \equiv V[\dot{\Phi}]$, where

$$V[\dot{\Phi}](x-y,t-s) = \theta(-\dot{\Phi}(x,t))\delta(x-y)\delta(t-s). \tag{28}$$

This enables us to calculate the higher order solutions by substituting Eq. (28) into Eq. (27) and by solving it successively order by order,

$$V_\varepsilon^{(n)} = V(\cdot\kappa\Delta_>''\cdot V)^{n-1}. \tag{29}$$

Thus, we have determined the functional $H_\varepsilon[\Phi]$ as an infinite power series with respect to g.

In passing, we briefly discuss the solution for F_ε. Expanding similarly the RG equation (22) in power series of g, we find that the solution of Eq. (22) is $F_\varepsilon=0$ because of the fact that the bare $F_{\varepsilon_0}=0$ as well as that the second term in the RHS of Eq. (22) is zero due to the trace with respect to the time variable.

Considering these, we end up with the renormalized action functional,

$$S_\varepsilon = \tilde{\Phi}\cdot\sum_{n=1}^\infty g^n V[\dot{\Phi}](\cdot\kappa\Delta_>''\cdot V[\dot{\Phi}])^{n-1}\cdot\kappa\Phi'' \equiv \sum_{n=1}^\infty g^n S_\varepsilon^{(n)} \tag{30}$$

with V defined by Eq. (28). We have thus determined the renormalized action in full order in g. This shows the efficiency of the present approach.

To obtain the asymptotic solution of the Barenblatt equation, we must set $J(x,t)=q\delta(x)\delta(t)$ to specify the initial condition and calculate dominant parts with respect to $\varepsilon/\varepsilon_0$ in the action obtained so far. The action decomposed into some sectors by the substitution of Eq. (17). Then, we readily notice that among them dominant contributions are from the $\tilde{\Phi}(x,t)=\tilde{\phi}_<(x,t)$ and $\Phi(x,t)=q\Delta(x,t)$ sectors in the asymptotic region $\varepsilon \ll t$. In the next section, we show that such calculations indeed lead to the anomalous diffusion exponent.

IV. ANOMALOUS DIFFUSION EXPONENT

In this section, we first give explicit calculations of the anomalous diffusion exponent at the leading orders with respect to g and check the renormalizability as well. Based on these calculations, we next present the formula for the higher order exponent.

A. First order

Let us start with the first order which Goldenfeld *et al.* have calculated within the perturbation in the iteration scheme. From Eq. (30), one can obtain

$$S_\varepsilon^{(1)} = q\int_0^\infty dt \int_{-\infty}^{\infty} dx \tilde{\phi}_<(x,t)\theta(-q\dot{\Delta}_>(x,t))\kappa\Delta''_>(x,t)$$
$$= -q\left(\frac{1}{\sqrt{2\pi e}}\ln\frac{\varepsilon}{\varepsilon_0}\right)\tilde{\phi}_<(0,0) + qS_{reg}, \tag{31}$$

where the regular part of $S^{(1)}$ is defined as

$$S_{reg}^{(1)} = \int_{\varepsilon_0}^{\varepsilon}\frac{dt}{t}\int_{-1}^{1} d\omega\, \delta\tilde{\phi}_<(\sqrt{2\kappa t}\omega,t)f(\omega) \tag{32}$$

with

$$\delta\tilde{\phi}(\sqrt{2\kappa t}\omega,t) \equiv \tilde{\phi}_<(\sqrt{2\kappa t}\omega,t) - \tilde{\phi}_<(0,0),$$

$$f(\omega) \equiv \frac{\omega^2-1}{2}\frac{e^{-\omega^2/2}}{\sqrt{2\pi}}. \tag{33}$$

Here, the regular term in Eq. (31) is not involved with the renormalization of the action since we can safely set $\varepsilon_0 \to 0$, while the first term is relevant to the renormalization of q, the height of the initial distribution. Thus it turns out that at this order q is indeed renormalized, whereas others, especially g, are not renormalized. This feature actually holds even in the next order, as will be checked below. This implies that the present system is always at a fixed point because g is not renormalized, and hence, the anomalous dimension which is in general a scheme-dependent quantity is a physical observable in the present case. Considering these, we introduce the renormalization only to q and define the renormalized q as $q_R = qZ$. Expanding the renormalization constant Z as

$$Z = 1 + \sum_{n=1} g^n Z^{(n)}, \tag{34}$$

with

$$Z^{(n)} = -\gamma^{(n)}\ln(\varepsilon/\varepsilon_0), \tag{35}$$

it turns out that the first order of Z reads

$$\gamma^{(1)} = \frac{1}{\sqrt{2\pi e}}, \tag{36}$$

from Eq. (31). This indeed reproduces the result of Goldenfeld *et al.* [13].

B. Second order

The renormalization of q introduced above is indeed enough to render the solution of the Barenblatt equation finite also in higher order. To verify this, let us next calculate the second order renormalized action. Due to similar arguments to the first order, the action (30) yields

$$S_\varepsilon^{(2)} = q\int_{2\varepsilon_0}^{\varepsilon_0+\varepsilon} dt \int_{-\sqrt{2\kappa t}}^{\sqrt{2\kappa t}} dx \int_{\varepsilon_0}^{t-\varepsilon_0} dt_1 \int_{-\sqrt{2\kappa t_1}}^{\sqrt{2\kappa t_1}} dx_1 \tilde{\phi}_<(x,t)\kappa\Delta''(x - x_1, t-t_1)\kappa\Delta''(x_1,t_1) + (\text{reg .}), \tag{37}$$

where reg. stands for regular parts of the renormalized action. A similar but rather lengthy calculation leads to

$$S_\varepsilon^{(2)} = q\frac{Z^{(1)2}}{2}\tilde{\phi}(0,0) + qZ^{(1)}S_{\text{reg}}^{(1)} + qZ^{(2)}\tilde{\phi}(0,0) + (\text{reg .}) \tag{38}$$

where $Z^{(2)} = -\gamma^{(2)}\ln(\varepsilon/\varepsilon_0)$ with

$$\gamma^{(2)} = -\int_0^1 \frac{d\tau_1}{\tau_1}\int_{-1}^1 d\omega_1 f(\omega_1) \times\left(\frac{1}{\sqrt{2\pi e}} - \frac{1-\sqrt{\tau_1}\omega_1}{1-\tau_1}\frac{e^{-(1-\sqrt{\tau_1}\omega_1)^2/[2(1-\tau_1)]}}{\sqrt{2\pi(1-\tau_1)}}\right). \tag{39}$$

This result indicates that the second order action correctly includes the contributions from the first order renormalized action; namely, the renormalized action with the source term $q\tilde{\phi}(0,0)$ satisfies $gS_\varepsilon^{(1)} + g^2 S_\varepsilon^{(2)} + q\tilde{\phi}(0,0) = e^Z qS_{reg} + e^Z q\tilde{\phi}(0,0)$ up to the second order of g. Therefore, we expect in general that

$$q_R = q(\varepsilon_0/\varepsilon)^\gamma, \tag{40}$$

where

$$\gamma = \sum_{n=1} g^n\gamma^{(n)}. \tag{41}$$

The constant γ thus obtained indeed gives the anomalous dimension of the solution for the present diffusion equation.

To see this explicitly, notice first that $u(x,t)$ is given by $u(x,t) = \delta\ln\mathcal{Z}/\delta\tilde{J}(x,t)$ and second that $\tilde{J}$ is not renormalized. Therefore, we have

$$u(x,t;q,\varepsilon_0) = u(x,t;q_R,\varepsilon), \tag{42}$$

which tells us that the solution is independent of ε. Hence, the renormalized solution should satisfy the following RG equation:

$$\left(\varepsilon\frac{\partial}{\partial\varepsilon} - \gamma q_R\frac{\partial}{\partial q_R}\right)u(x,t;q_R,\varepsilon) = 0, \tag{43}$$

where γ is defined by Eq. (41), or alternatively by

$$\gamma = -\partial \ln q_R/\partial \ln \varepsilon. \tag{44}$$

On the other hand, the dimensional analysis requires

$$u(x,t;q,\varepsilon_0) = q/\sqrt{\kappa t}\Phi(x/\sqrt{\kappa t},\varepsilon_0/t). \tag{45}$$

Therefore, combining these observations, we can assume

$$u(x,t;q_R,\varepsilon) = q_R/\sqrt{\kappa t}\Phi(x/\sqrt{\kappa t},\varepsilon/t). \tag{46}$$

Substituting this into Eq. (43), it turns out that the relation

$$\Phi(x/\sqrt{\kappa t},\varepsilon/t) = (\varepsilon/t)^{\gamma}\tilde{\Phi}(x/\sqrt{\kappa t}) \tag{47}$$

holds. Hence, the asymptotic behavior of the solution for the Barenblatt equation is indeed given by

$$u \sim 1/t^{1/2+\gamma}. \tag{48}$$

So far we have derived the anomalous diffusion exponent up to second order with respect to g. The exponent of first order is the same as that obtained by Goldenfeld *et al.*, whereas the exponent of second order (39) is estimated as $\gamma^{(2)} \sim -0.063\,546$ by numerical integration. Although this value is different from the result in Ref. [13], it coincides with that obtained later by Cole and Wagner given by a different integral formula derived via a different method [18]. Since the anomalous diffusion exponent should be scheme-independent in the present case, we believe that our result as well as Cole and Wagner's is correct.

C. Higher order

The efficiency of our method lies in the fact that one can compute the higher order exponent in a similar way as above. To present the exponent of nth order, we define a function

$$g(\omega_1,\omega_2,\tau_2) = \frac{e^{-(\omega_1-\sqrt{\tau_2}\omega_2)^2/[2(1-\tau_2)]}}{\sqrt{2\pi(1-\tau_2)}} - \frac{e^{-\omega_2^2/2}}{\sqrt{2\pi}}. \tag{49}$$

Then Eq. (30) yields

$$\gamma^{(n)} = -\int_0^1 \prod_{j=1}^{n-1}\frac{d\tau_j}{\tau_j}\int_{-1}^1 \prod_{j=1}^{n-1} d\omega_j f(\omega_{n-1}) \times\left(\frac{1}{\sqrt{2\pi e}} - \frac{1-\sqrt{\tau_1}\omega_1}{1-\tau_1}\frac{e^{-(1-\sqrt{\tau_1}\omega_1)^2/[2(1-\tau_1)]}}{\sqrt{2\pi(1-\tau_1)}}\right) \times\prod_{j=1}^{n-2}\frac{1}{2}\frac{d^2 g(\omega_j,\omega_{j+1},\tau_{j+1})}{d\omega_j^2}. \tag{50}$$

For reference, we numerically estimate the exponent of third order, $\gamma^{(3)} = -0.003\,14$. The exponent obtained so far seems a good convergent series.

V. SUMMARY AND DISCUSSION

In this paper, we have applied ERG techniques to the initial-value problem of the Barenblatt equation, one of the typical nonlinear diffusion equations. We have derived the anomalous diffusion exponent in full order with respect to the parameter controlling the nonlinearity. This implies that the ERG approach is efficient for systems far from equilibrium described by nonlinear partial-differential equations as well as for field theories and statistical mechanics.

Although we have been able to obtain the formula of the diffusion exponent given by multiple integrals, it seems interesting to explore it in a nonperturbative way, since the formula of $\gamma^{(n)}$ obtained in this paper is still hard to compute numerically for large n. To this end, we need to develop methods of solving nonperturbatively the ERG equation for the nonlinear diffusion equations presented in this paper.

The present approach would be useful for other types of initial-value problem for nonlinear diffusion equations. In particular, application to the critical dynamic of nonlinear traveling waves is of great interest. One of the well known examples is the Kolmogorov-Pitrovsky-Piscounov equation [19] which shows an interesting universal behavior in the selection of the front velocity. This problem has been addressed by Paquette *et al.* [20], but more detailed analysis is needed if one wants to understand, e.g., the universal logarithmic corrections to the velocity in the pulled front, from the RG point of view. Application of the ERG to such problems would be quite interesting.

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